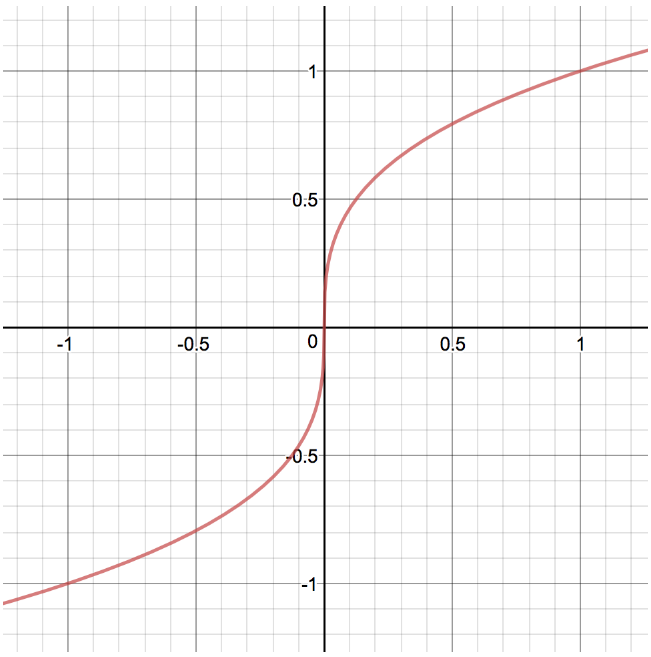
Worksheet on Differentiation

True of False.

Problem 1. Consider the function $f(x) = x^{1/3}$.

1. The function $f(x)$ is continuous at 0.
2. The function $f(x)$ is differentiable at 0.
3. The function $f(x)$ has a tangent line at $(0, 0)$.
4. Sketch the graph $y = f(x)$

Solution:
1. True.
2. False, the limit $\lim_{h \to 0} \frac{h^{1/3}}{h}$ does not exist.
3. True, there exists the tangent line at 0. But it is vertical and so its slope is undefined. That's why the derivative does not exist.
4.

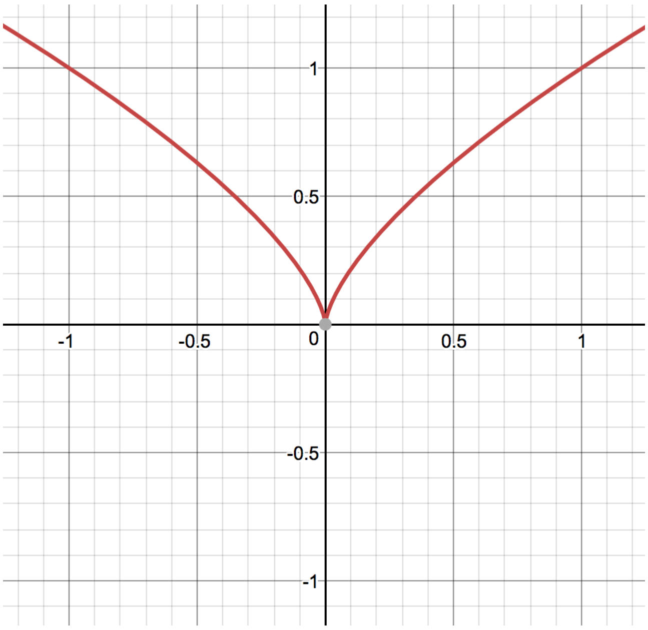


Problem 2. Consider the function $f(x) = x^{2/3}$.

1. The function $f(x)$ is continuous at 0.
2. The function $f(x)$ is differentiable at 0.
3. The function $f(x)$ has a tangent line at $(0, 0)$.
4. Sketch the graph $y = f(x)$

Solution:
1. True.
2. False.
3. False – there is a cusp at $x = 0$ and therefore there is no tangent line.

4.



Problem 3. If $f'(a)$ exists then $\lim_{x \to a} f(x)$

1. must exist, but there is not enough information to determine it
2. equals $f(a)$
3. equals $f'(a)$
4. may not exist

Solution: The answer is 2. If f is differentiable at $x = a$, it must be continuous at $x = a$, and so the limit does exist and equals $f(a)$.

Problem 4. Compute the following derivatives.

1. $f(x) = 2^{2016}$

   Solution: the derivative is 0 since f is constant.

2. $f(x) = \frac{1}{16}x^4$

   Solution: using the power rule, $f'(x) = \frac{1}{4}x^3$

3. $f(x) = x^2(5-2x)$

   Solution: using the product rule, $f'(x) = 2x(5-2x) + x^2 \cdot (-2) = -6x^2 + 10x$.

4. $f(x) = \sqrt{x} - x$

   Solution: $f'(x) = \frac{1}{2\sqrt{x}} - 1$.

5. $f(x) = e^{\sqrt{2x+1}}$

   Solution: using the chain rule, $f'(x) = e^{\sqrt{2x+1}} \cdot \frac{1}{2\sqrt{2x+1}} \cdot 2 = \frac{e^{\sqrt{2x+1}}}{\sqrt{2x+1}}$

6. $f(x) = \sin x \cos x$

   Solution: one can use the product rule, or notice that $f(x) = \frac{1}{2}\sin(2x)$ and so using the chain rule $f'(x) = \frac{1}{2}\cos(2x) \cdot 2 = \cos(2x)$.

7. $f(x) = \cot x$

   Solution: using the quotient rule we get $f'(x) = -\frac{1}{\sin^2 x}$.

8. Suppose the derivative of $\ln x$ exists. Find it using the chain rule. (Hint: use $e^{\ln a} = a$.)

   Solution: we know $e^{\ln x} = x$. Taking derivatives on both sides and using the chain rule we get $e^{\ln x} \cdot (\ln x)' = 1$, and so $(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$.

9. Using the previous problem, show that the derivative of x^r is rx^{r-1} for *any real number* r.

   Solution:

   We have $x^r = e^{\ln x^r} = e^{r \ln x}$. Computing the derivative using the chain rule, we get

$$(x^r)' = e^{r\ln x} \cdot r(\ln x)' = r\frac{e^{r\ln x}}{x} = r\frac{x^r}{x} = rx^{r-1}$$